

# RICCI FLOW AND QUANTUM THEORY

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ABSTRACT. We show how Ricci flow is related to quantum theory via Fisher information and the quantum potential.

## 1. INTRODUCTION

In [9, 13, 14] we indicated some relations between Weyl geometry and the quantum potential, between conformal general relativity (GR) and Dirac-Weyl theory, and between Ricci flow and the quantum potential. We would now like to develop this a little further. First we consider simple Ricci flow as in [35, 49]. Thus from [35] we take the Perelman entropy functional as **(1A)**  $\mathfrak{F}(g, f) = \int_M (|\nabla f|^2 + R) \exp(-f) dV$  (restricted to  $f$  such that  $\int_M \exp(-f) dV = 1$ ) and a Nash (or differential) entropy via **(1B)**  $N(u) = \int_M u \log(u) dV$  where  $u = \exp(-f)$  ( $M$  is a compact Riemannian manifold without boundary). One writes  $dV = \sqrt{\det(g)} \prod dx^i$  and shows that if  $g \rightarrow g + sh$  ( $g, h \in \mathcal{M} = \text{Riem}(M)$ ) then **(1C)**  $\partial_s \det(g)|_{s=0} = g^{ij} h_{ij} \det(g) = (\text{Tr}_g h) \det(g)$ . This comes from a matrix formula of the form **(1D)**  $\partial_s \det(A + B)|_{s=0} = (A^{-1} : B) \det(A)$  where  $A^{-1} : B = a^{ij} b_{ji} = a^{ij} b_{ij}$  for symmetric  $B$  ( $a^{ij}$  comes from  $A^{-1}$ ). If one has Ricci flow **(1E)**  $\partial_s g = -2\text{Ric}$  (i.e.  $\partial_s g_{ij} = -2R_{ij}$ ) then, considering  $h \sim -2\text{Ric}$ , one arrives at **(1F)**  $\partial_s dV = -R dV$  where  $R = g^{ij} R_{ij}$  (more general Ricci flow involves **(1G)**  $\partial_t g_{ik} = -2(R_{ik} + \nabla_i \nabla_k \phi)$ ). We use now  $t$  and  $s$  interchangeably and suppose  $\partial_t g = -2\text{Ric}$  with  $u = \exp(-f)$  satisfying  $\square^* u = 0$  where  $\square^* = -\partial_t - \Delta + R$ . Then  $\int_M \exp(-f) dV = 1$  is preserved since **(1H)**  $\partial_t \int_M u dV = \int_M (\partial_s u - Ru) dV = -\int_M \Delta u dV = 0$  and, after some integration by parts,

$$(1.1) \quad \partial_t N = \int_M [\partial_t u (\log(u) + 1) dV + u \log(u) \partial_t dV] = \int_M (|\nabla f|^2 + R) e^{-f} dV = \mathfrak{F}$$

In particular for  $R \geq 0$   $N$  is monotone as befits an entropy. We note also that  $\square^* u = 0$  is equivalent to **(1I)**  $\partial_t f = -\Delta f + |\nabla f|^2 - R$ .

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It was also noted in [49] that  $\mathfrak{F}$  is a Fisher information functional (cf. [8, 10, 24, 25]) and we showed in [13] that for a given 3-D manifold  $M$  and a Weyl-Schrödinger picture of quantum evolution based on [42, 43] (cf. also [4, 5, 6, 8, 9, 10, 11, 12, 16, 17, 51]) one can express  $\mathfrak{F}$  in terms of a quantum potential  $Q$  in the form  $(1J) \mathfrak{F} \sim \alpha \int_M Q P dV + \beta \int_M |\vec{\phi}|^2 P dV$  where  $\vec{\phi}$  is a Weyl vector and  $P$  is a probability distribution associated with a quantum mass density  $\hat{\rho} \sim |\psi|^2$ . There will be a corresponding Schrödinger equation (SE) in a Weyl space as in [10, 13] provided there is a phase  $S$  (for  $\psi = |\psi| \exp(iS/\hbar)$ ) satisfying  $(1K) (1/m) \text{div}(P \nabla S) = \Delta P - R P$  (arising from  $\partial_t \hat{\rho} - \Delta \hat{\phi} = -(1/m) \text{div}(\hat{\rho} \nabla S)$  and  $\partial_t \hat{\rho} + \Delta \hat{\rho} - R \hat{\rho} = 0$  with  $\hat{\rho} \sim P \sim u \sim |\psi|^2$ ). In the present work we show that there can exist solutions  $S$  of  $(1K)$  and this establishes a connection between Ricci flow and quantum theory (via Fisher information and the quantum potential). Another aspect is to look at a relativistic situation with conformal perturbations of a 4-D semi-Riemannian metric  $g$  based on a quantum potential (defined via a quantum mass). Indeed in a simple minded way we could perhaps think of a conformal transformation  $\hat{g}_{ab} = \Omega^2 g_{ab}$  (in 4-D) where following [14] we can imagine ourselves immersed in conformal general relativity (GR) with metric  $\hat{g}$  and  $(1L) \exp(Q) \sim \mathfrak{M}^2/m^2 = \Omega^2 = \hat{\phi}^{-1}$  with  $\beta \sim \mathfrak{M}$  where  $\beta$  is a Dirac field and  $Q$  a quantum potential  $Q \sim (\hbar^2/m^2 c^2)(\square_g \sqrt{\rho})/\sqrt{\rho}$  with  $\rho \sim |\psi|^2$  referring to a quantum matter density. The theme here (as developed in [14]) is that Weyl-Dirac action with Dirac field  $\beta$  leads to  $\beta \sim \mathfrak{M}$  and is equivalent to conformal GR (cf. also [8, 10, 36, 45, 46, 47] and see [28] for ideas on Ricci flow gravity).

**REMARK 1.1.** For completeness we recall (cf. [10, 50]) for  $\mathfrak{L}_G = (1/2\chi)\sqrt{-g}R$

$$(1.2) \quad \delta \mathfrak{L} = \frac{1}{2\chi} \left[ R_{ab} - \frac{1}{2} g_{ab} R \right] \sqrt{-g} \delta g^{ab} + \frac{1}{2\chi} g^{ab} \sqrt{-g} \delta R_{ab}$$

The last term can be converted to a boundary integral if certain derivatives of  $g_{ab}$  are fixed there. Next following [7, 9, 14, 27, 38, 39, 40] the Einstein frame GR action has the form

$$(1.3) \quad S_{GR} = \int d^4x \sqrt{-g} (R - \alpha (\nabla \psi)^2 + 16\pi L_M)$$

(cf. [7]) whose conformal form (conformal GR) is

$$(1.4) \quad \hat{S}_{GR} = \int d^4x \sqrt{-\hat{g}} e^{-\psi} \left[ \hat{R} - \left( \alpha - \frac{3}{2} \right) (\hat{\nabla} \psi)^2 + 16\pi e^{-\psi} L_M \right] =$$

$$\int d^4x \sqrt{-g} \left[ \hat{\phi} \hat{R} - \left( \alpha - \frac{3}{2} \right) \frac{(\hat{\nabla} \hat{\phi})^2}{\hat{\phi}} + 16\pi \hat{\phi}^2 L_M \right]$$

where  $\hat{g}_{ab} = \Omega^2 g_{ab}$ ,  $\Omega^2 = \exp(\psi) = \phi$ , and  $\hat{\phi} = \exp(-\psi) = \phi^{-1}$ . If we omit the matter Lagrangians, and set  $\lambda = (3/2) - \alpha$ , (1.4) becomes for  $\hat{g}_{ab} \rightarrow g_{ab}$

$$(1.5) \quad \tilde{S} = \int d^4x \sqrt{-g} e^{-\psi} [R + \lambda(\nabla\psi)^2]$$

In this form on a 3-D manifold  $M$  we have exactly the situation treated in [10, 13] with an associated SE in Weyl space based on **(1K)**. ■

## 2. SOLUTION OF **(1K)**

Consider now **(1K)**  $(1/m)\text{div}(P\nabla S) = \Delta P - RP$  for  $P \sim \hat{\rho} \sim |\psi|^2$  and  $\int P\sqrt{|g|}d^3x = 1$  (in 3-D we will use here  $\sqrt{|g|}$  for  $\sqrt{-g}$ ). One knows that  $\text{div}(P\nabla S) = P\Delta S + \nabla P \cdot \nabla S$  and

$$(2.1) \quad \Delta\psi = \frac{1}{\sqrt{|g|}}\partial_m(\sqrt{|g|}\nabla\psi); \quad \nabla\psi = g^{mn}\partial_n\psi; \quad \int_M \text{div}\mathbf{V}\sqrt{|g|}d^3x = \int_{\partial M} \mathbf{V} \cdot \mathbf{ds}$$

(cf. [10]). Recall also  $\int P\sqrt{|g|}d^3x = 1$  and

$$(2.2) \quad Q \sim -\frac{\hbar^2}{8m} \left[ \left( \frac{\nabla P}{P} \right)^2 - 2 \left( \frac{\Delta P}{P} \right) \right]; \quad \langle Q \rangle_\psi = \int PQd^3x$$

Now in 1-D an analogous equation to **(1K)** would be **(3A)**  $(PS')' = P' - RP = F$  with solution determined via

$$(2.3) \quad PS' = P' - \int RP + c \Rightarrow S' = \partial_x \log(P) - \frac{1}{P} \int RP + cP^{-1} \Rightarrow \\ \Rightarrow S = \log(P) - \int \frac{1}{P} \int RP + c \int P^{-1} + k$$

which suggests that solutions of **(1K)** do in fact exist in general. We approach the general case in Sobolev spaces à la [1, 2, 15, 22]. The volume element is defined via  $\eta = \sqrt{|g|}dx^1 \wedge \dots \wedge dx^n$  (where  $n = 3$  for our purposes) and  $*$  :  $\wedge^p M \rightarrow \wedge^{n-p} M$  is defined via

$$(2.4) \quad (*\alpha)_{\lambda_{p+1}\dots\lambda_n} = \frac{1}{p!} \eta_{\lambda_1\dots\lambda_n} \alpha^{\lambda_1\dots\lambda_p}; \quad (\alpha, \beta) = \frac{1}{p!} \alpha_{\lambda_1\dots\lambda_p} \beta^{\lambda_1\dots\lambda_p};$$

$$*1 = \eta; \quad **\alpha = (-1)^{p(n-p)}\alpha; \quad *\eta = 1; \quad \alpha \wedge (*\beta) = (\alpha, \beta)\eta;$$

One writes now  $\langle \alpha, \beta \rangle = \int_M (\alpha, \beta)\eta$  and, for  $(\Omega, \phi)$  a local chart we have **(2A)**  $\int_M f dV = \int_{\phi(\Omega)} (\sqrt{|g|}f) \circ \phi^{-1} \prod dx^i$  ( $\sim \int_M f \sqrt{|g|} \prod dx^i$ ). Then one has **(2B)**  $\langle d\alpha, \gamma \rangle = \langle \alpha, \delta\gamma \rangle$  for  $\alpha \in \wedge^p M$  and  $\gamma \in \wedge^{p+1} M$  where the codifferential  $\delta$  on p-forms is defined via **(2C)**  $\delta = (-1)^p *^{-1} d*$ . Then  $\delta^2 = d^2 = 0$  and  $\Delta = d\delta + \delta d$  so that  $\Delta f = \delta df = -\nabla^\nu \nabla_\nu f$ . Indeed for  $\alpha \in \wedge^p M$

$$(2.5) \quad (\delta\alpha)_{\lambda_1, \dots, \lambda_{p-1}} = -\nabla^\gamma \alpha_{\gamma, \lambda_1, \dots, \lambda_{p-1}}$$

with  $\delta f = 0$  ( $\delta : \wedge^p M \rightarrow \wedge^{p-1} M$ ). Then in particular **(2D)**  $\langle \Delta \phi, \phi \rangle = \langle \delta d\phi, \phi \rangle = \langle d\phi, d\phi \rangle = \int_M \nabla^\nu \phi \nabla_\nu \phi \eta$ .

Now to deal with weak solutions of an equation in divergence form look at an operator **(2E)**  $Au = -\nabla(a\nabla u) \sim (-1/\sqrt{|g|})\partial_m(\sqrt{|g|}ag^{mn}\nabla_n u) = -\nabla_m(a\nabla^m u)$  so that for  $\phi \in \mathcal{D}(M)$

$$(2.6) \quad \begin{aligned} \int_M Au\phi dV &= - \int [\nabla_m(ag^{mn}\nabla_n u)]\phi dV = \\ &= \int ag^{mn}\nabla_n u \nabla_m \phi dV = \int a\nabla^m u \nabla_m \phi dV \end{aligned}$$

Here one imagines  $M$  to be a complete Riemannian manifold with Soblev spaces  $H_0^1(M) \sim H^1(M)$  (see [1, 3, 15, 26, 29, 48]). The notation in [1] is different and we think of  $H^1(M)$  as the space of  $L^2$  functions  $u$  on  $M$  with  $\nabla u \in L^2$  and  $H_0^1$  means the completion of  $\mathcal{D}(M)$  in the  $H^1$  norm  $\|u\|^2 = \int_M [|u|^2 + |\nabla u|^2]dV$ . Following [29] we can also assume  $\partial M = \emptyset$  with  $M$  connected for all  $M$  under consideration. Then let  $H = H^1(M)$  be our Hilbert space and consider the operator  $A(S) = -(1/m)\nabla(P\nabla S)$  with

$$(2.7) \quad B(S, \psi) = \frac{1}{m} \int P \nabla^m S \nabla_m \psi dV$$

for  $S, \psi \in H_0^1 = H^1$ . Then  $A(S) = RP - \Delta P = F$  becomes **(2F)**  $B(S, \psi) = \langle F, \psi \rangle = \int F\psi dV$  and one has **(2G)**  $|B(S, \psi)| \leq c\|S\|_H \|\psi\|_H$  and  $|B(S, S)| = \int P(\nabla S)^2 dV$ . Now  $P \geq 0$  with  $\int P dV = 1$  but to use the Lax-Milgram theory we need here  $|B(S, S)| \geq \beta\|S\|_H^2$  ( $H = H^1$ ). In this direction one recalls that in Euclidean space for  $\psi \in H_0^1(\mathbf{R}^3)$  there follows **(2H)**  $\|\psi\|_{L^2}^2 \leq c\|\nabla\psi\|_{L^2}^2$  (Friedrich's inequality - cf. [48]) which would imply  $\|\psi\|_H^2 \leq (c+1)\|\nabla\psi\|_{L^2}^2$ . However such Sobolev and Poincaré-Sobolev inequalities become more complicated on manifolds and **(2H)** is in no way automatic (cf. [1, 29, 48]). However we have some recourse here to the definition of  $P$ , namely  $P = \exp(-f)$ , which basically is a conformal factor and  $P > 0$  unless  $f \rightarrow \infty$ . One heuristic situation would then be to assume **(2I)**  $0 < \epsilon \leq P(x)$  on  $M$  (and since  $\int \exp(-f)dV = 1$  with  $dV = \sqrt{|g|} \prod_1^3 dx^i$  we must then have  $\epsilon \int dV \leq 1$  or  $\text{vol}(M) = \int_M dV \leq (1/\epsilon)$ ). Then from **(2G)** we have **(2J)**  $|B(S, S)| \geq \epsilon\|(\nabla S)^2\|$  and for any  $\kappa > 0$  it follows that  $|B(S, S)| + \kappa\|S\|_{L^2}^2 \geq \min(\epsilon, \kappa)\|S\|_{H^1}^2$ . This means via Lax-Milgram that the equation

$$(2.8) \quad A(S) + \kappa S = -\frac{1}{m}\nabla(P\nabla S) + \kappa S = F = RP - \Delta P$$

has a unique weak solution  $S \in H^1(M)$  for any  $\kappa > 0$  (assuming  $F \in L^2(M)$ ). Equivalently **(2K)**  $-\frac{1}{m}[P\Delta S + (\nabla P)(\nabla S)] + \kappa S = F$  has a unique weak solution  $S \in H^1(M)$ . This is close but we cannot put  $\kappa = 0$ .

A different approach following from remarks in [29], pp. 56-57 (corrected in [30], p. 248), leads to an heuristic weak solution of **(1K)**. Thus from a result of Yau [53] if  $M$  is a complete simply connected 3-D differential manifold with sectional curvature  $K < 0$  one has for  $u \in \mathcal{D}(M)$

$$(2.9) \quad \int_M |\psi| dV \leq (2\sqrt{-K})^{-1} \int_M |\nabla \psi| dV \Rightarrow \int_M |\psi|^2 dV \leq c \int_M |\nabla \psi|^2 dV$$

Hence **(2H)** holds and one has  $\|\psi\|_{H^1}^2 \leq (1+c)\|\nabla \psi\|^2$ . Moreover if  $M$  is bounded and simply connected with a reasonable boundary  $\partial M$  (e.g. weakly convex) one expects **(2L)**  $\int_M |\psi|^2 dV \leq c \int_M |\nabla \psi|^2 dV$  for  $\psi \in \mathcal{D}(M)$  (cf. [41]. In either case **(2M)**  $|B(S, S)| \geq \epsilon \|(\nabla S)^2\| \geq (c+1)^{-1} \epsilon \|S\|_{H_0^1}^2$  and this leads via Lax-Milgram again to a sample result

**THEOREM 2.1.** Let  $M$  be a bounded and simply connected 3-D differential manifold with a reasonable boundary  $\partial M$ . Then there exists a unique weak solution of **(1K)** in  $H_0^1(M)$ .

**REMARK 2.1.** One must keep in mind here that the metric is changing under the Ricci flow and assume that estimates involving e.g.  $K$  are considered over some time interval. ■

**REMARK 2.2.** There is an extensive literature concerning eigenvalue bounds on Riemannian manifolds and we cite a few such results. Here  $I_\infty(M) \sim \inf_\Omega (A(\partial\Omega)/V(\Omega))$  where  $\Omega$  runs over (connected) open subsets of  $M$  with compact closure and smooth boundary (cf. [18, 19]). Yau's result is  $I_\infty(M) \geq 2\sqrt{-K}$  (with equality for the 3-D hyperbolic space) and Cheeger's result involves  $\|\nabla \phi\|_{L^2} \geq (1/2)I_\infty(M)\|\phi\|_{L^2} \geq \sqrt{-K}\|\phi\|_{L^2}$ . There are many other results where e.g.  $\lambda_1 \geq c(Vol(M))^{-2}$  for  $M$  a compact 3-D hyperbolic manifold of finite volume (see [21, 34, 44] for this and variations). There are also estimates for the first eigenvalue along a Ricci flow in [33, 37] and estimates of the form  $\lambda_1 \geq 3K$  for closed 3-D manifolds with Ricci curvature  $R \geq 2K$  ( $K > 0$ ) in [32, 33]. In fact Ling obtains  $\lambda_1 \geq K + (\pi^2/\tilde{d}^2)$  where  $\tilde{d}$  is the diameter of the largest interior ball in nodal domains of the first eigenfunction. There are also estimates  $\lambda_1 \geq (\pi^2/d^2)$  ( $d = \text{diam}(M)$ ,  $R \geq 0$ ) in [31, 52, 54] and the papers of Ling give an excellent survey of results, new and old, including estimates of a similar kind for the first Dirichlet and Neumann eigenvalues. ■

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